APPLICATIONS OF ELLIPTIC OPERATOR METHODS TO C^{∞}- CONVERGENCE PROBLEM

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Dedicated to Prof. Romulus Cristescu on his 70th birthday

1. Introduction

The aim of this paper is to extend classical results in Complex Analysis to the general setting of elliptic operators. Clearly, this idea goes back to the mathematicians of the XIXth century, who developed a parallel between analytic functions of one variable and harmonic functions in planar domains. Nowadays, Hörmander's book on several complex variables [5] demonstrates the importance of $\overline{\partial}-$techniques in Complex Analysis and thus supports the point of view that Analysis of differential operators represents the right way for extending most classical results. See also [6] and [11].

Our investigation here is restricted to the problem of extending the classical results due to K. Weierstrass, G. Vitali, W. F. Osgood et al. on $C^{\infty}-$convergence of sequences of analytic functions.

Further consequences to Bergman Space Theory will be presented elsewhere.

2. Preliminaries on elliptic operators

Throughout this section Ω will denote a bounded open subset of $\mathbb{R}^N$ and $C^{\infty}(\Omega, r)$ will denote the Fréchet space $C^{\infty}(\Omega, \mathbb{C}^r)$, endowed with the family of seminorms

$$||u||_n^K = \sum_{|\alpha|\,\leq\, n} \frac{1}{\alpha!} \sup_{x\in K} |D^{\alpha}u(x)|\,,$$

where n runs over $\mathbb{N}$ and K runs over the compact subsets of Ω.

We shall consider linear elliptic operators $P : C^{\infty}(\Omega, r) \to C^{\infty}(\Omega, s)$ of order m, i.e. operators of the form

$$(Pu)(x) = \sum_{|\alpha|\,\leq\, m} a_{\alpha}(x)(D^{\alpha}u)(x)$$

whose leading symbols

$$\sigma_P(x, \xi) = \sum_{|\alpha|\,=\,m} a_{\alpha}(x)\xi^{\alpha} : \mathbb{C}^r \to \mathbb{C}^s$$

are injective, whenever $x \in \Omega$ and $\xi \in \mathbb{R}^r \setminus \{0\}$; all coefficients are supposed to be C^{∞}.

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The simplest examples of linear elliptic operators are

$$\frac{d^p}{dx^p}, \quad \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}, \quad \Delta^p$$

and their perturbations by lower order terms.

Much of the theory of elliptic operators depends upon the powerful methods of Functional Analysis and in this connection an important role is played by Sobolev spaces.

For $m \in \mathbb{N}$, the Sobolev space $H^m(\Omega, r)$ is the space of all functions $u \in L^2(\Omega, r) = L^2(\Omega, \mathbb{C}^r)$, whose distributional derivatives of order $\leq m$ are in $L^2(\Omega, r)$. This is a Hilbert space for the norm

$$||u||_{H^m(\Omega,r)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^2 \, dx\right)^{1/2}.$$

Notice that $H^k(\Omega, r)$ can be described alternatively as the completion of

$$\left\{u \in C^\infty(\Omega, r); \ ||u||_{H^m(\Omega,r)} < \infty\right\}$$

with respect to $||\cdot||_{H^k(\Omega,r)}$.

According to Sobolev embedding theorem [9], if $k > \frac{N}{2} + j$, then for every compact subset K of Ω there exists a constant $C_1 > 0$ such that

$$||u||_j^K \leq C_1 \, ||u||_{H^k(\Omega,r)}$$

whenever $u \in C^\infty(\Omega, r)$. This theorem shows that every $u \in H^m(\Omega, r)$ is a. e. equal to a function of class $C^{m-[N/2]-1}$.

A basic result on linear elliptic operators is the Friedrichs' inequality [4]: *If P is as above, then for every relatively compact open subset Ω' of Ω there exists a constant $C_2 > 0$ such that*

$$||u||_{H^{m+k}(\Omega',r)} \leq C_2 \left(||Pu||_{H^k(\Omega,r)} + ||u||_{H^0(\Omega,r)}\right)$$

for every $k \in \mathbb{N}$ and every $u \in C^\infty(\Omega, r)$ for which the right hand side is finite.

Elliptic operators of the type considered above have nice regularity properties. Particularly they are *hypoelliptic,* i.e. if $u \in L^2_{loc}(\Omega, r)$ and $Pu = 0$ (in the sense of distributions), then u is a.e. equal to a $C^\infty-$function.

3. $C^\infty-$ CONVERGENCE OF SEQUENCES OF $P-$ ANALYTIC FUNCTIONS

As above, $P : C^\infty(\Omega, r) \to C^\infty(\Omega, s)$ will denote a linear elliptic operator of order m. Attached to it will be the vector space

$$\mathcal{A}(P) = Ker\, P$$

of the so called $P-$ *analytic* functions. The usual analytic functions correspond to the case where $P = \overline{\partial}$. For $P = \frac{d^p}{dx^p}$, $\mathcal{A}(P)$ consists of all polynomials of degree $\leq p$, while for $P = \Delta$ we retrieve the case of harmonic functions.

The convergence of sequences of $P-$analytic functions has some special features, noticed in particular cases by a number of mathematicians such as K. Weierstrass, G. Vitali, H. Harnack et al. A sample of the results in this area is Weierstrass' theorem, which asserts that uniform convergence on compacta preserves analyticity.

In order to develop a unifying approach in the framework of $P-$analyticity, we have to make the following basic remark, which combines Friedrichs' inequality and Sobolev embedding theorem:

Lemma 3.1. *Let $(u_n)_n$ be a sequence of elements of $C^{\infty}(\Omega, r)$ such that:*
i) $(Pu_n)_n$ is a converging sequence in $C^{\infty}(\Omega, s)$;
ii) $\lim\limits_{j,\, k\to\infty} \int_K |u_k - u_j|^2 \, dx = 0$
for every compact subset K *of* Ω.
Then $(u_n)_n$ is a converging sequence in $C^{\infty}(\Omega, r)$.

Lemma 3.1 yields a number of criteria of $C^{\infty}-$ convergence, which provide themselves very useful in concrete applications:

Corollary 3.2. (*Vitali's Criterion of C^{∞} − convergence*). *Suppose that $(u_n)_n$ is a sequence of $P-$analytic functions such that:*
i) $(u_n)_n$ is pointwise convergent to a function $u : \Omega \to \mathbb{C}^r$;
ii) $(u_n)_n$ is uniformly bounded on each compact subset of Ω.
Then u is $P-$analytic and $u_n \to u$ in $C^{\infty}(\Omega, r)$.

Proof. Use the theorem of Lebesgue on dominated convergence. ■

Corollary 3.3. (*Weierstrass' Criterion of C^{∞} − convergence*). *If $(u_n)_n$ is a sequence of $P-$analytic functions and $u_n \to u$ uniformly on each compact subset of Ω, then u is $P-$analytic and $u_n \to u$ in $C^{\infty}(\Omega, r)$.*

The discussion above shows that $\mathcal{A}(P)$ constitutes a Fréchet space (and also a closed subspace of $C^{\infty}(\Omega, r)$) when endowed with the family of seminorms

$$||u||_K = \sup_{x\in K} |u(x)|$$

where K runs over the compact subsets of Ω.

Theorem 3.4. (*Stieltjes-Vitali Criterion of Compactness*). *Every sequence of $P-$analytic functions which is bounded on compacta contains a converging subsequence.*

Proof. First notice that Ω can be represented as the union of an increasing sequence of compact subsets e.g., $\Omega = \bigcup\limits_{n=1}^{\infty} \Omega_n$ where

$$\Omega_n = \{x \in \Omega;\ |x| \leq n \text{ and } dist\,(x, \partial\Omega) \geq 1/n\}$$

for each $n \in \mathbb{N}^{\star}$.

By the Sobolev embedding theorem, we get uniform estimates for the derivatives of the u_n's on each subset Ω_n. In particular, the functions u_n are equicontinuous. By the Arzela-Ascoli theorem we can choose a uniformly converging subsequence on each Ω_n and using a diagonal argument we obtain a subsequence converging uniformly an each compact subset of Ω.

To end the proof it remains to apply to that subsequence the result of Corollary 3.3 above. ■

As a consequence of Theorem 3.4 we obtain that condition i) in Vitali's criterion of $C^{\infty}-$convergence can be weakned as:

i′) $(u_n(x))_n$ is convergent for x in a dense subset of Ω.

How far is pointwise convergence from $C^{\infty}-$convergence in the case of $P-$analytic functions? The answer is given by the following theorem, which extends a result due to W. F. Osgood [7]:

Theorem 3.5. *Let $(u_n)_n$ be a sequence of $P-$analytic functions which is pointwise converging to a function $u : \Omega \to \mathbb{C}^r$. Then u is $P-$analytic in a dense open subset $\Omega_1 \subset \Omega$ and convergence is uniform on compact subsets of Ω_1.*

Proof. Let K be an arbitrary closed ball included in Ω. Then $K = \bigcup_{n=1}^{\infty} K_n$, where the K_n's are closed subsets defined as

$$K_n = \{x \in K : |u_k(x)| \leq n \text{ for every } k\}.$$

By the Baire category theorem some K_m must have non-empty interior. For this m the sequence $(u_n)_n$ is uniformly bounded on $Int\, K_m$, hence by Corollary 3.2 above it converges uniformly on compact subsets of $Int\, K_m$. Thus u is $P-$analytic on $Int\, K_m$. Since the argument can be applied to any closed ball, it follows that u is $P-$analytic on a dense open subset $\Omega_1 \subset \Omega$. The fact that the convergence is uniform on compacta contained in Ω_1 is standard and we omit the details. ■

A natural question arising in connection with Theorem 3.4 is how *thin* can the subset Ω_1 be? One can prove easily that for each open subset Ω of $\mathbb{R}^N$ and each $\varepsilon > 0$ there must exist a dense open subset $\Omega_\varepsilon \subset \Omega$ whose Lebesgue measure is $< \varepsilon$. The problem is how to fit the convergence aspects as in Theorem 3.4.

The following example could be useful to settle that problem. Let $\lambda \in (0, 1/2)$. From the closed unit square $K_0 = [0,1] \times [0,1]$ delete $[0,1] \times (\lambda, 1-\lambda) \cup (\lambda, 1-\lambda) \times [0,1]$, thus leaving a set K_1 of 4 closed squares. Continue in a similar manner so that at the nth stage we are left with a set K_n of 4^n closed squares, whose centers we denote $z_{n,k}$ $(k = 1, ..., 4^n)$. Then $K_\infty = \bigcap_n K_n$ is a totally disconnected set, of planar Lebesgue measure zero. Letting

$$f(z) = \lim_{n\to\infty} \frac{1}{4^n} \sum_{k=1}^{4^n} \frac{1}{z - z_{n,k}}$$

we obtain a function continuous on K_0, which has no analytic continuation off $K_0 \setminus K_\infty$.

Notice that the Hausdorff dimension of the exceptional set K_∞ is $-\log 4/\log \lambda$, a quantity which goes to 2 as $\lambda \to 1/2$.

The example above shows that the implication

$$u = \text{continuous } \& \ P(u) = 0 \text{ a.e. } \Rightarrow P(u) = 0 \text{ everywhere}$$

fails even for $P = \overline{\partial}$. However, for $P = \overline{\partial}$ one can prove the following result on removable singularities:

Theorem 3.6. (*A. S. Besicovitch* [1]) *. If Ω is an open subset of $\mathbb{C}$ and $u : \Omega \to \mathbb{C}$ is a continuous function such that $\overline{\partial} u = 0$ except on a thin subset, then $\overline{\partial} u = 0$ everywhere i.e., u is analytic.*

Recall that a subset of $\mathbb{R}^N$ is called *thin* if it has $\sigma-$finite $(N-1)-$dimensional Hausdorff measure.

Conjecture 3.7. *Theorem 3.6 extends to all elliptic operators.*

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